Thermodynamic fluctuations and the fluctuation-imposed limit for temperature measurement

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In experimental sciences, random processes often place a fundamental limit on the achievable resolution. A well known example is the Johnson noise voltage across a resistor. In this paper, we describe the observation of temperature fluctuations in a thermometer caused by the random transfer of heat in and out of it. This process places a fundamental limit on the resolution of a thermometer. We find that the noise of the thermometer at low frequency is given by $4Rk_BT^2$ (in units of K^2/Hz), where R is the thermal resistance that 1 inks the sensing element to the object whose temperature is to be measured. This implies that for noise reduction purposes, R is the only available engineering parameter to adjust. In a recent thermometer design, we have minimized R to achieve a resolution of 5x $10^{-11} K/\sqrt{Hz}$.

In thermodynamic terms, a *thermometer* is modeled as a subsystem in contact with a heat reservoir. The *reservoir,* the object whose temperature is to be measured, is assumed to be isothermal and to have infinite heat capacity. In equilibrium thermodynamics, thermodynamic properties of the subsystem are known to undergo spontaneous fluctuations even in equilibrium. The aim of a thermometer is to reconstruct the temperature of the reservoir from measurements of a temperature dependent quantity of the subsystem. The accuracy of temperature measurements is therefore limited by the fluctuations of the quantity as a result of random energy transfer between the subsystem and the reservoir. In a macroscopic system, equilibrium thermodynamics tells us that fractional fluctuations go approximately as $1/\sqrt{N}$, where N is the number of particles. N is perhaps 1022 for a typical thermometer, therefore causing a fractional fluctuations of the order of 10-^{1}1. Although this value is very small, it is within the range of resolution of state-of-the-art thermometry with reported resolution of the order of 10^{-9} to 10-'0 $K/\sqrt{Hz}$ [1]. In a recent paper[2], the noise spectrum from one such thermometer was measured and was found to agree very well with the fluctuation dissipation theorem (FDT). The FDT is an extension of equilibrium thermodynamics that includes time dependent fluctuations. In an experimental setup not much different from an ordinary incandescent light bulb, the intensity of light from a hot filament was observed to fluctuate with an amplitude consistent with the theory of thermodynamic fluctuations. In the following, we describe results of two experiments which further clarify the mechanism which limits the resolution of temperature measurement.

The high resolution thermometer (HRT) used in both experiments employed a paramagnetic salt, copper ammonium bromide [$Cu(NH_4)_2Br_4{\cdot}2H_2O$] (C-AH), which has a Curie temperature near 1.8 K. The change of the magnetization of the salt induced by variations of temperature is measured in a constant magnetic field using a superconducting quantum interference device (SQUID). A superconducting tube is used to supplied a very stable DC field for the measurement. The SQUID voltage output is calibrated against a germanium resistance thermometer. This kind of thermometer has been routinely used in the studies of phase transitions[4] near the superfluid transition temperature of helium, 2.177 K.

In the first experiment reported here we used a thermometer consisting of a single salt pill which was thermally linked to a superfluid helium reservoir. Superfluid helium is an ideal substance for construction of an almost ideal thermal reservoir with a fast internal thermal response time due to its large heat capacity and a high thermal conductivity below the lambda point. The thermal control system used in this experiment is very similar to onc reported previously

The solid dot in figure 1 shows the measured RMS temperature noise of the thermometer. From equilibrium thermodynamics, the RMS noise is given by

$$\langle(\Delta T)^2\rangle = k_B T^2 / C, \qquad (1)$$

where C is the heat capacity of the salt pill. The heat capacity C is obtained by scaling to the data of Miedema et al.[5,6]. The estimated temperature noise using Equ. (1) is the shaded band in figure 1. There is no adjustable parameters in this plot except for the scaling uncertainty of ±1 O%. The inset of the figure shows the histograms of ΔT for two reservoir temperatures along with the Gaussian function fit used to find the width of the probability distribution of fluctuations. This result indicates that the noise of the system originated from the spontaneous temperature fluctuations of the thermometer. The very good agreement between noise data and the fit to the equation (1) confirms the conclusion in Ref. [2] that the change in the mean energy (AU) and the change in the mean temperature (ΔT) in the fundamental relation AU = CAT can be treated as *the width of the distribution around its mean* with reasonable accuracy.

To gain a deeper understanding of the thermodynamic fluctuations, we have constructed another thermometer consisting of two salt pills which are connected to each other. Figure 2 shows a schematic drawing of this HRT with two CAB crystals of similar size. The two crystals arc connected together with a strong thermal link, with a weaker link made to a thermal reservoir. The temperature of each crystal was measured independently by separate SQUID magnetometers. This allows us to trace the flow of energy from one salt crystal to the other. As this happens, the conservation of energy implies that when the temperature of one crystal rises the temperature of the other must fall.

After the temperature of the thermal reservoir is stabilized to 1.891 K, the temperatures of the two salt pills are independently recorded. The Fourier transforms of temperature noises (T_1, and T_2) are computed and normalized in a way that $(T_1)_f \cdot (T_1)_f^*$ gives the power spectral density (PSD) of T_1, where $(T_1)_f$ is the normalized Fourier transform of T_1.

The open circles in figure 3 show the real part of the cross-correlation function, $\mathrm{Re}[(T_1)_f \cdot (T_2)_f^*]$, computed from the data. A negative value in this plot indicates anti-correlation. At lower frequencies the temperatures in the two HRTs are correlated while they become anti-correlated at higher frequencies. This can be qualitatively explained by the fact that the thermal link to the heat reservoir was designed to be much weaker than the link between the HRTs, resulting in low frequency energy fluctuations between a HRT and the reservoir and high frequency fluctuations between the HRTs. We have also measured the cross correlation between one of the salt crystal in the double thermometers with the single thermometer in a different location but attached to the same reservoir. The signal between them is completely uncorrelated, consistent with the above interpretation.

The thermal model shown in the inset of the figure 3 gives the following results based on FDT[7] combined with the conservation of energy:

$$(T_1)_f \cdot (T_2)_f^* = \left(\frac{4k_BT^2}{C_1 + C_2}\right) \cdot \left[\frac{\tau_0}{(1 + 4\pi^2\tau_0^2 f^2)} - \frac{\tau_{eff}}{(1 + 4\pi^2\tau_{eff}^2 f^2)}\right] \qquad (2)$$

where $\tau_o = R_o(C_1 + C_2)$, R_0 is the thermal resistance between the reservoir and subsystems, $C_{1,2}$ arc the heat capacities of salt 1 and 2, $\tau_{eff} = R_{12}C_{eff}$. R_{12} is the thermal resistance of the link between the salts, and $C_{eff}=C_1C_2/(C_1+C_2)$. This expression for the cross-correlation function is given by a combination of a correlated part and an anti correlated part. The area under the curve of the correlated part is equal to the area under the anti-correlated part, A fit to this function gives $(C_1 + C_2) = 15$ mJ/K, $\tau_o = 2.3$ see, $\tau_{eff} = 0.5$ sec. From the heat capacity data of Meidema et al., we estimated that $(C_1 + C_2) = 10 \pm 2$ mJ/K, in reasonable agreement with the fit.

The equation for thermodynamic fluctuations, $(\Delta T)^2 = k_BT^2/C$ suggests that the temperature resolution may be increased by increasing the heat capacity of the thermometer. However increasing C also increases the time constant (τ) of the measurement. Because it is only possible to get one statistically independent reading per time interval τ, longer τ means fewer data points are available with which to determine the average of the Gaussian distribution of observed readings, which is the real temperature of the system. Since the error in the average varies as $1/\sqrt{N_o}$ where NO is the number of data points, we expect that the thermometry noise goes as $\sqrt{\tau} \bullet \delta T \approx \sqrt{\tau}/\sqrt{C} = \sqrt{R}$. By decreasing R, more thermodynamic averaging over the fluctuating energy states takes place, yielding higher resolution thermometry. Quantitatively, the fluctuation-dissipation theorem gives the noise power density as $(T)_f.(T)_f^* = 4R\, k_B T^2$ at low frequencies, where the thermometer is usable. Therefore, R is the only relevant parameter that matters in the construction of an ultra-high resolution thermometer. It implies that detector noise and salt sensitivity will <u>never</u> be a problem !

To this end, an HRT using gadolinium trichloride, $GdCl_3$, salt was constructed with the aim of keeping R as low as possible. A drawing of the new HRT is shown in the inset of figure 4. The thermal link to the salt was accomplished by machining numerous chimney-shaped heat fins into a high purity copper rod. The machining was accomplished by wire electro-discharge machining. The space between the heat fins was filled with the salt material by slow crystallization in a quartz tube filled with the molten salt. The thermal link was designed not only to decrease the thermal impedance, but to decrease the noise pickup due to spontaneously fluctuating electrical currents in the metal, an effect which was found to be proportional to $A\sqrt{N_F}$, where N_F is the number of chimney shaped fins and *A is* the cross sectional area of the fin.

The noise spectrum of the thermometer at 2. 17K is shown in figure 4. The figure of merit of the thermometer is the noise level at very low frequency, because that is the rate at which the error in the measurement decreases at long integration times. We find for the $GdCl_3$ thermometer, at low frequencies, the resolution is 5x10-1] K/ {Hz. We expect that further increases in thermometer resolution may be realized by further minimizing R.

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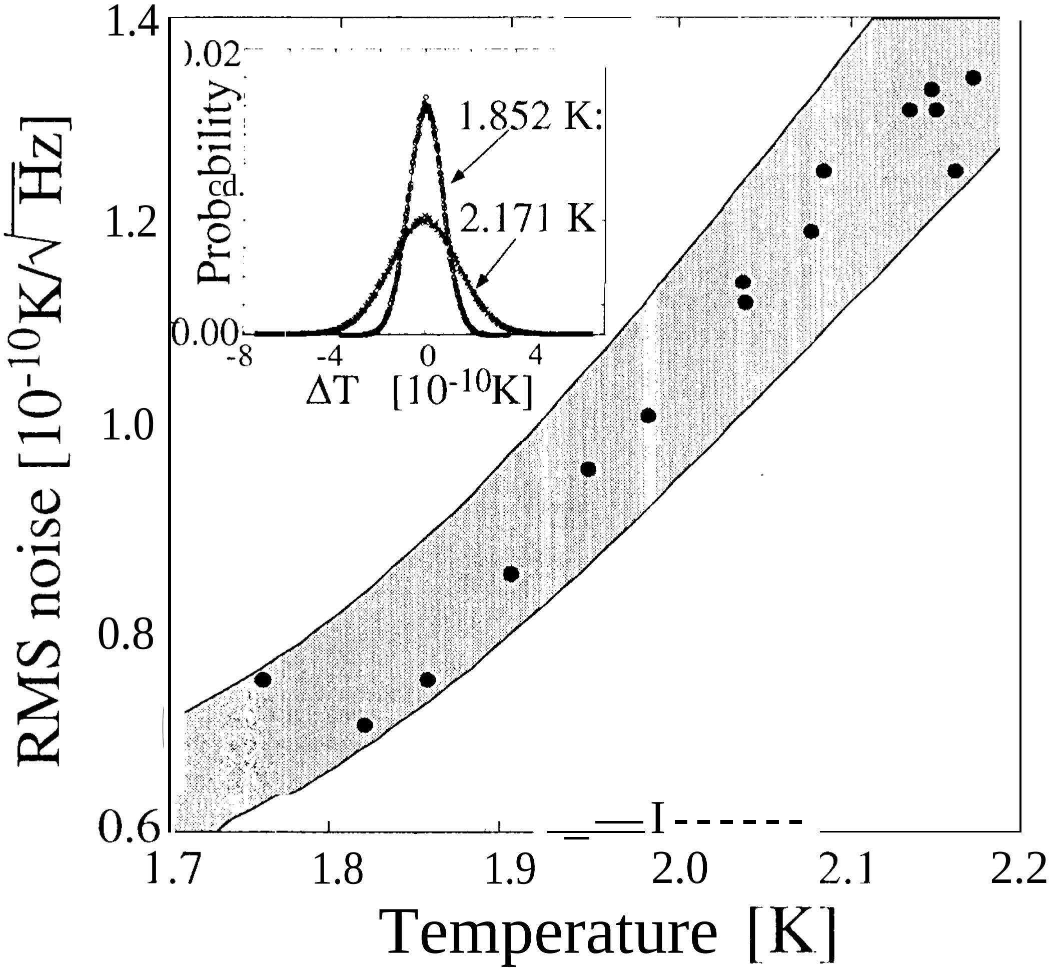
Figure Captions

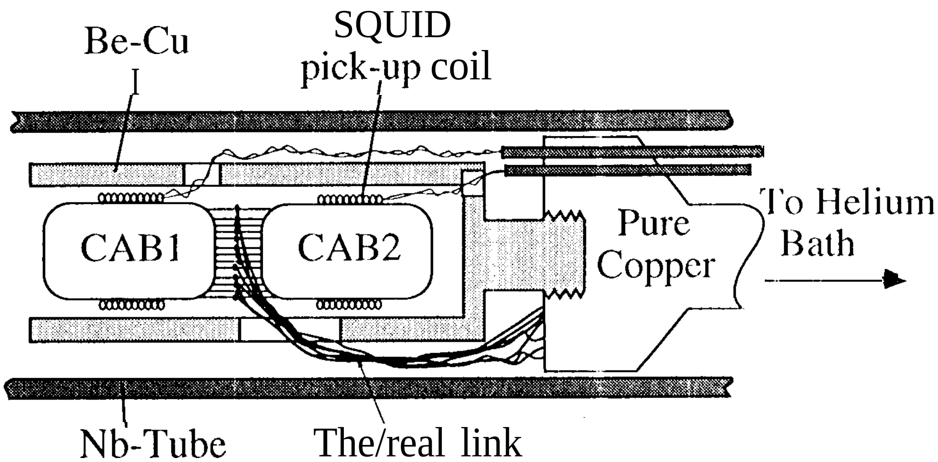
Fig. 1: Temperature dependence of temperature noise. Inset: Histograms of the outputs of an HRT at two different reservoir temperatures. The solid lines are gaussian fits. The width of the gaussian function represents the fluctuation of temperature variable around its mean value,

Fig.2: Schematic drawing of double HRTs.

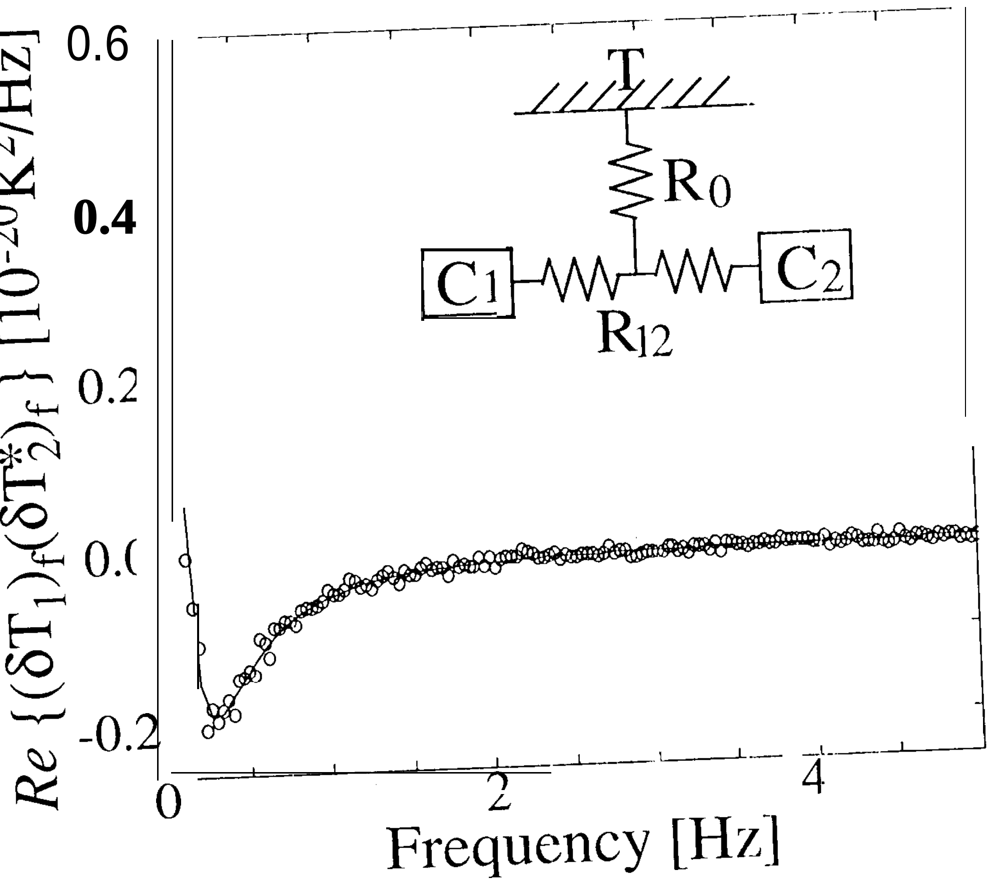
Fig.3: Cross-power spectrum of the outputs from the double HRTs. Inset: the thermal model of the double thermometer.

Fig.4: Noise spectrum from the $GdCl_3$ thermometer. Inset: a drawing of the $GdCl_3$ thermometer.

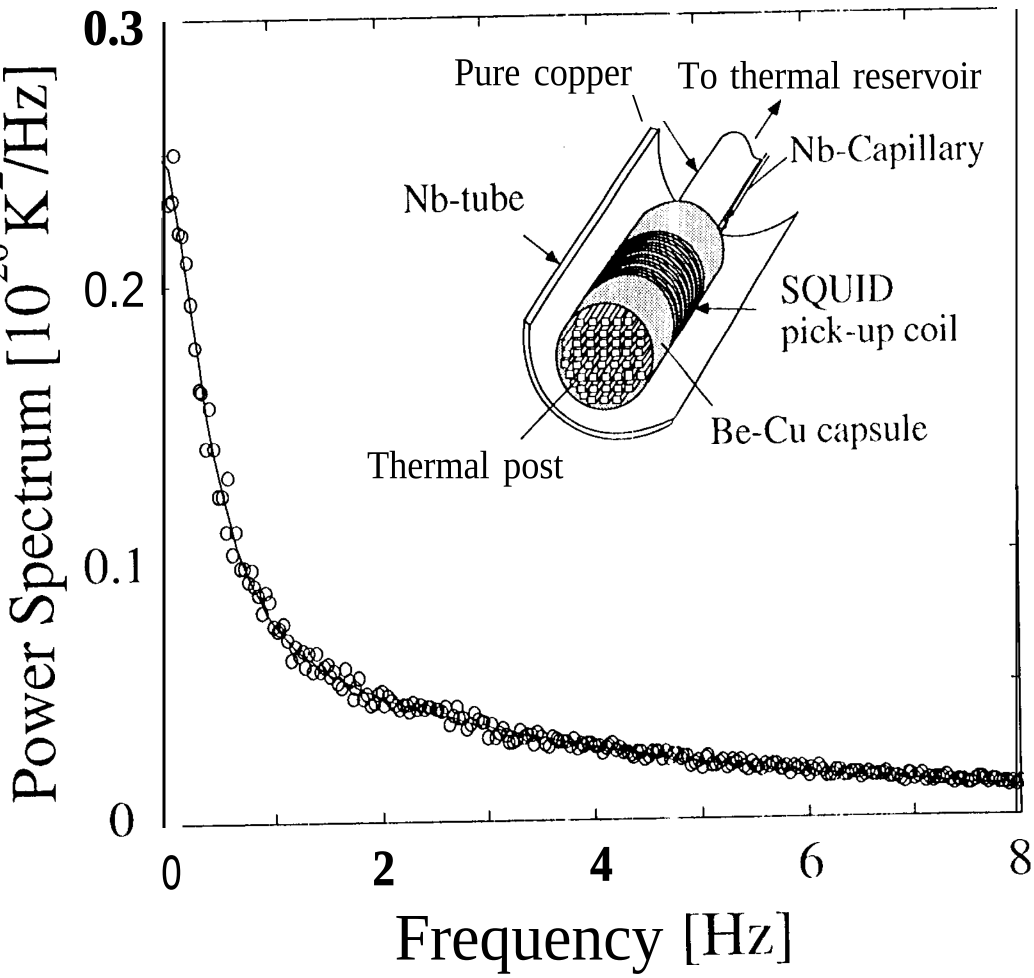

RMS noise [10^{-10}K/$\sqrt{\text{Hz}}$]
Temperature [K]
1.4
1.2
1.0
0.8
0.6
1.7
1.8
1.9
2.0
2.1
2.2
Probability
0.02
0.00
1.852 K
2.171 K
-8
-4
0
4
ΔT [10^{-10}K]



Be-Cu
SQUID
pick-up coil
CAB1
CAB2
Pure
Copper
To Helium
Bath
Nb-Tube
The/real link



T
R0
C1
R12
C2
0.6
0.4
0.2
0.0
-0.2
Re {(δT1)f(δT*2)f} [10-20K2/Hz]
0
2
4
Frequency [Hz]


Pure copper
To thermal reservoir
Nb-Capillary
Nb-tube
SQUID
pick-up coil
Be-Cu capsule
Thermal post
Power Spectrum [10^{-20} K^2/Hz]
0.3
0.2
0.1
0
0
2
4
6
8
Frequency [Hz]